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Informal Report

C.3

M-3 Series on Detonation Science

II. One-Dimensional Flow

University of California

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II. One-Dimensional Flow

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M-3 SERIES ON DETONATION SCIENCE

II. ONE-DIMENSIONAL FLOW

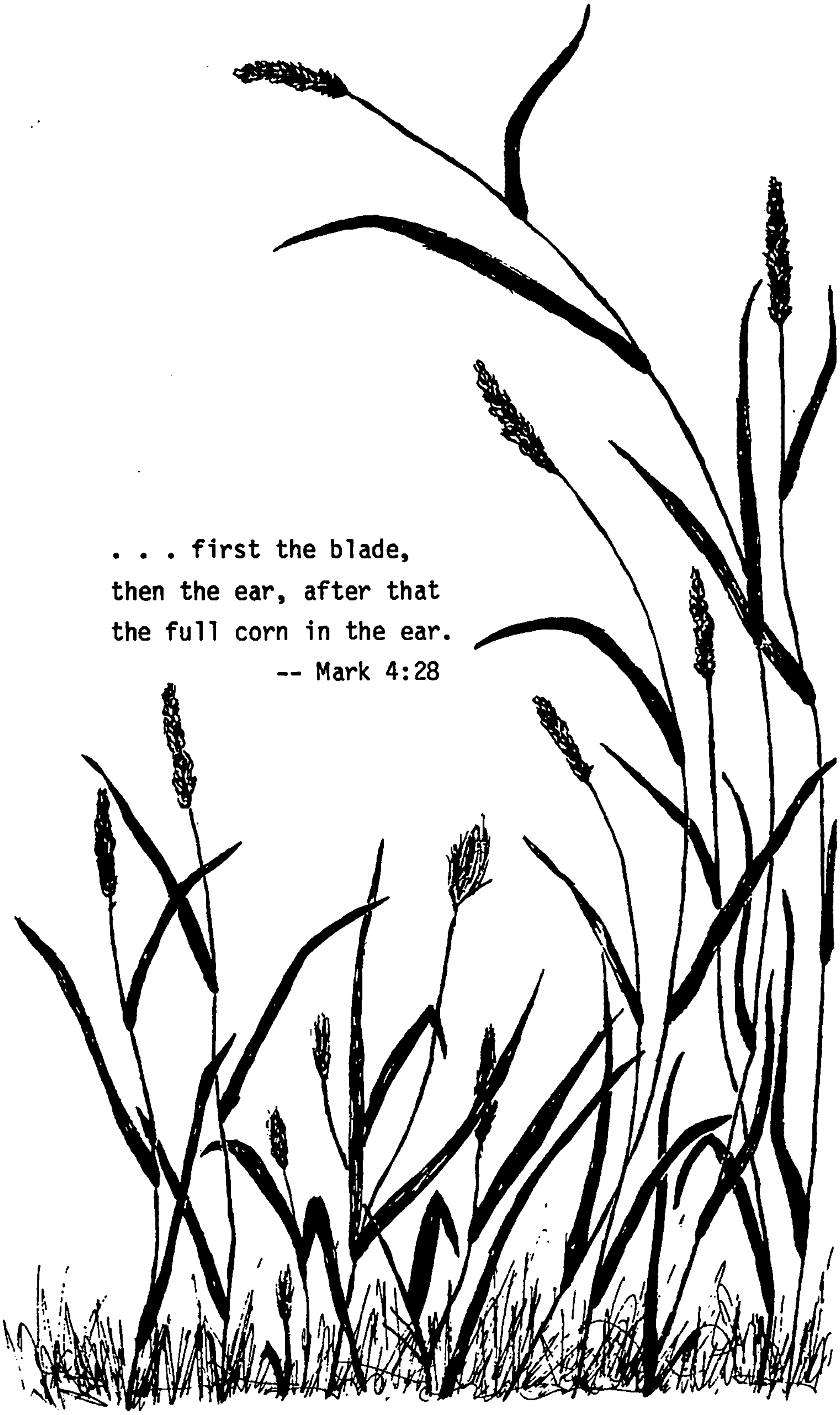
by

Wildon Fickett

ABSTRACT

The M-3 Series on Detonation Science consists of lectures on various topics in detonation theory and explosives research. The plan is to have each set of lectures accompanied by notes in the form of a LAMS report.

The first set of lectures, by W. C. Davis, offered an overview; the report for these has not been written. The second set, for which this report constitutes the notes, is an introduction to one-dimensional nonreactive flow. The plan of attack is to, in effect, take the equations apart and put them back together again, adding complications one by one as the reassembly proceeds.



. . . first the blade,
then the ear, after that
the full corn in the ear.
-- Mark 4:28

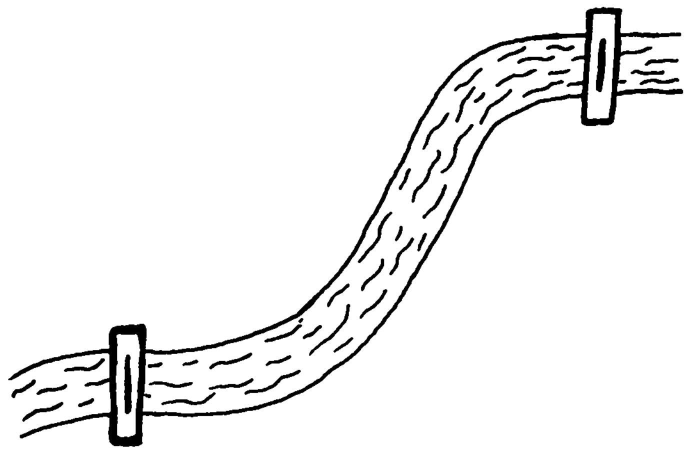
I. INTRODUCTION

. . . again the Physics foreshadows the solution.

-- J. Hadamard, Lectures on Cauchy's Problem, Ch. 2.

Keywords

conservation
dissipation
equation types:
linear/nonlinear
hyperbolic/other
kinematic equation of state
kinematic waves
linearization



The governing equations were given in the first set of lectures of this series, hereafter referred to as I. They are

$$\dot{\rho} + \rho u_x = 0 \quad (1.1a)$$
$$\dot{u} + v p_x = 0 \quad (1.1b)$$
$$\dot{e} + p\dot{v} = 0 \quad (1.1c)$$
$$e = e(p,v) \quad (1.1d)$$
$$v \equiv 1/\rho \quad .$$

The first three equations express the conservation of mass, momentum, and energy, and the fourth is the equation of state (EOS) characteristic of the material. Throughout we shall require that the EOS be well-behaved, that is that

$$p_e > 0$$
$$p_\rho > 0$$
$$p_{\rho\rho} > 0 \quad .$$

The dot is shorthand for a derivative along a particle path

$$\dot{f} \equiv f_t + u f_x \quad .$$

The only dissipative mechanism in the system is the shock wave, treated as a jump discontinuity. The Rankine-Hugoniot conditions governing this jump, which are algebraic relations relating the states on the two sides of the discontinuity to its propagation velocity, are derivable from the differential equations. The passage of a fluid element through a shock is an irreversible process which raises its entropy. the entropy jump can be calculated from the Rankine-Hugoniot conditions if the (thermally) complete EOS is known.

Our object is to understand the general properties of these equations and the qualitative nature of their solutions, with particular reference to large-amplitude waves in bodies made up of layers of different materials.

The equations are nonlinear (the coefficients of the derivatives depend on the dependent variables) and hyperbolic (their solutions are propagating waves). Our approach is to begin with the simplest mathematical object having some of their properties, and then add new features one at a time until we finally arrive at the full set of equations.

The simplest such mathematical object is the law of conservation of mass expressed by the first equation (1.1a). This is perhaps more transparent in integral form. If x_1 and x_2 are two fixed stations in a time-varying one-dimensional flow, the conservation of mass is expressed by

$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)dx = \rho u(x_1,t) - \rho u(x_2,t) \quad , \tag{1.2}$$

which states that the time rate of change of the mass between the two stations x_1 and x_2 is equal to the difference in the fluxes through the boundaries. Taking the time derivative inside the integral on the left and passing to the limit $x_1 \to x_2$ gives the partial differential equation

$$\rho_t + (\rho u)_x = 0 \quad , \tag{1.3}$$

identical with (1.1a).

We would like this to be our only equation of motion. But it has two unknowns: ρ and u. So we <u>assume</u> a simple (mathematical) "equation of state"

$$u = u(\rho) \quad , \tag{1.4}$$

so that (1.3) and (1.4) together constitute a determinate system for ρ(or u). The usual (equivalent) formulation is to give the mass flux ρu a name and let it be a function of ρ. For reasons which will become clear later, we choose p as the symbol for mass flux

$$p \equiv \rho u \qquad (1.5)$$
$$p = p(\rho) \quad ,$$

so that our system of equations for $\rho(x,t)$ is

$$\rho_t + p_x = 0 \qquad (1.6a)$$
$$p = p(\rho) \quad . \qquad (1.6b)$$

In this context p is <u>not</u> the physical pressure but <u>is</u> a <u>mathematical analog</u> of it.

The system (1.6) is called a <u>kinematic wave equation</u>, because it often applies in good approximation to those situations, such as the extremely slow flow of a glacier, in which the kinematics (fluid distortion) of the flow is important, but the dynamics (momentum and energy changes) are not.

We have now got down to just one equation of motion plus an equation of state with a single independent variable. But our differential equation is still nonlinear, for, using (1.6b) we can write (1.6a) as

$$\rho_t + c(\rho)\rho_x = 0 \qquad (1.7)$$
$$c(\rho) \equiv p'(\rho) \quad ,$$

in which the coefficient $c(\rho)$ of the space derivative ρ_x is a function of the dependent variable ρ. (We choose the symbol c for $p'(\rho)$ knowing that it will turn out to be the wave speed.)

To make things still simpler we <u>linearize</u> (1.7) about a constant state $\rho = \overline{\rho}$. The resulting equation will describe small-amplitude disturbances propagating in this constant state. Before performing the linearization we change notation slightly. Let $\hat{\rho}$ be the density, expressed as the sum of the unperturbed density $\overline{\rho}$ and a perturbation $\epsilon\rho$, with ϵ small, that is

$$\hat{\rho}(x,t) = \overline{\rho} + \epsilon\rho(x,t) \quad . \qquad (1.8)$$

We take $\overline{\rho}$ and ρ of the same order, letting the order parameter ε express the order of the perturbation. Substituting (1.8) into (1.7) gives

$$(\overline{\rho} + \varepsilon\rho)_t + (\overline{c} + \overline{c}'\varepsilon\rho + \dots)(\overline{\rho} + \varepsilon\rho)_x = 0 \quad ,$$

(in which $c(\rho)$ has been expanded in a Taylor's series about $\overline{\rho}$) and where

$$\overline{c} = c(\overline{\rho}) = p'(\overline{\rho}) \quad , \quad c' = c'(\overline{\rho}) = p''(\overline{\rho}) \quad .$$

Multiplying out and retaining only terms of order ε, we obtain the first-order equation for the perturbation $\rho(x,t)$

$$\rho_t + \overline{c}\rho_x = 0 \quad . \tag{1.9}$$

This single, linear, first-order partial differential equation is our starting point for the next chapter. We now suspend further use of our perturbation notation until then.

Problems

1. Problem 1.1. Traffic Flow. Take a continuum model for single-lane traffic flow in one direction with ρ the number of cars per unit length. Suppose that each driver at all times sets his own speed instantaneously to some function of the local density, so that we have the required function $u = u(\rho)$.

Write down the function $p(\rho)$ and the differential equation (1.7) for two cases (the significance of which will appear later)

(1) Safe drivers, $u(\rho) = a$ = constant.

(2) Reckless drivers, $u(\rho) = 1/2\ \rho$.

2. Problem 1.2. Thermodynamics. Our experiments usually determine the function $e(p,v)$. Suppose that this function is given; write a partial differential equation for the unknown function $T(p,v)$.

Hint: The equation sought turns out to be a linear kinematic wave equation (in log T)

$$a(p,v)T_p + b(p,v)T_v = T \quad ,$$

with the coefficients calculated from the given function $e(p,v)$. There are different ways of deriving this. One way is to start with the differential of e

$$de = e_p dp + e_v dv$$

and the thermodynamic identity

$$(\partial e/\partial v)_T = T(\partial p/\partial T)_v - p \quad .$$

3. Problem 1.3. Flood Waves. Consider a thin layer of incompressible fluid of density ρ^* flowing slowly down a rectangular channel of unit width inclined slightly at an angle α to the horizontal, Fig. 1.1. Assume that the flow can be described in the one-dimensional approximation, that is, with everything a function of x and t only, with x distance along the channel.

Let h be the (normal) height of the fluid above the bottom of the channel. The quantity ρ to be used in the kinematic wave equation is a generalized density, the mass per unit area

$$\rho = \rho^* h \quad .$$

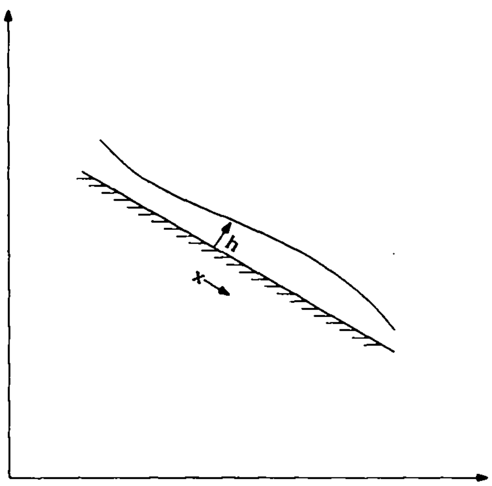


Fig. 1.1. Flood waves.

The body forces on an element of length dx (mass ρdx) are: gravity

$$\rho g \sin \alpha \, dx \quad ,$$

and bottom friction, assumed proportional to u^2

$$ku^2 dx \quad ,$$

with friction coefficient k. Obtain the kinematic equation of state by equating these two forces, and write the kinematic wave equation with h as the independent variable.

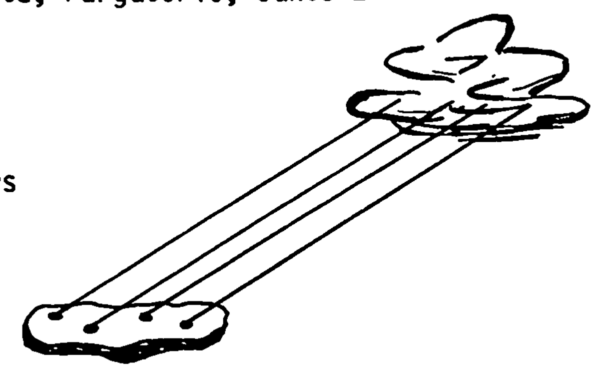
Note that equating the gravity and friction forces is equivalent to adding terms representing these to the right side of the momentum equation (1b) and neglecting the derivative terms in it.

II. ONE LINEAR EQUATION

> . . .like travelers pondering the road ahead
> who send their souls on while their bodies delay.
> -- Dante, Purgatorio, Canto 2

Keywords

- boundary conditions
- characteristics - information carriers
- characteristic speed (sound speed)
- directional derivative
- parametric solutions



We start with the simplest possible system: a single linear equation in one unknown. The results are very simple, but lay the groundwork for the later elaborations as we gradually build up to the full system.

The solutions are just those of the (physical) acoustic equation, except that they represent waves propagating in only one direction. There is a single family of characteristics, along which information is propagated.

A. Differential Equations

We treat the linear equation with one independent variable, the linearized kinematic wave equation (1.9) of the preceding chapter

$$\rho_t + \bar{c}\rho_x = 0 \quad , \tag{2.1}$$

where we have returned to the perturbation notation used there.

B. Characteristics

The most important property of hyperbolic equations is that they posses characteristics -- paths in the independent-variable space along which the equations take on a particularly simple form.

The key idea here is that of the directional derivative. Suppose we are given a function f(x,t), defined over the t-x plane. It has partial derivatives f_t and f_x, which we may think of as total derivatives of f along the special directions parallel to the tand x-axes, respectively. We want to find the total derivative of f along some arbitrary given direction (unrelated to f). This derivative can be expressed as a linear combination of f_t and f_x, just as a vector can be expressed as a linear combination of its components. Consider an arbitrary monotone curve C in t-x with slope

$$(dx/dt)_C = m(t) \quad .$$

Let us calculate the total derivative of f with respect to t along this curve. We can do this from the total differential of f

$$df = f_t dt + f_x dx \quad .$$

The desired derivative is

$$\begin{aligned} (df/dt)_C &= f_t + (dx/dt)_C f_x \\ &= f_t + m(t) f_x \quad . \end{aligned}$$

At each point of the curve this is the derivative of f with respect to t in the direction m, the tangent to the curve, or the directional derivative of f with respect to t in the direction m.

With this result in mind, we see that our differential equation (2.1) is just the directional derivative of ρ in the (constant) direction c. We can thus write it in the equivalent form

$$d\rho/dt = 0 \quad \text{on} \quad dx/dt = \bar{c} \quad . \tag{2.2}$$

Thus along any member of the one-parameter family of straight lines with slope $\bar{c}$, the partial differential equation (2.1) becomes an ordinary differential equation and is thus much easier to solve. The curves $dx/dt = \bar{c}$ are the characteristics and (2.2) is the characteristic form of (2.1). In this case, the partial differential equation (2.1) is essentially in characteristic form as given. In general, as we shall see in the next chapter, putting the equations into characteristic form requires some effort.

C. Boundary Conditions

A particular solution to (2.1) is defined by a boundary condition -- a set of boundary (or initial) data, consisting of specified values of ρ along some noncharacteristic arc like BB of Fig. 2.1.

Now the characteristic equations (2.2) state that ρ is constant along any line of slope $\bar{c}$. We see from the figure that the given boundary data -- values of ρ along the noncharacteristic arc BB -- define the solution everywhere between the characteristics C_1 and C_2 through its end points. To satisfy the boundary conditions, the constant value of ρ on each characteristic must

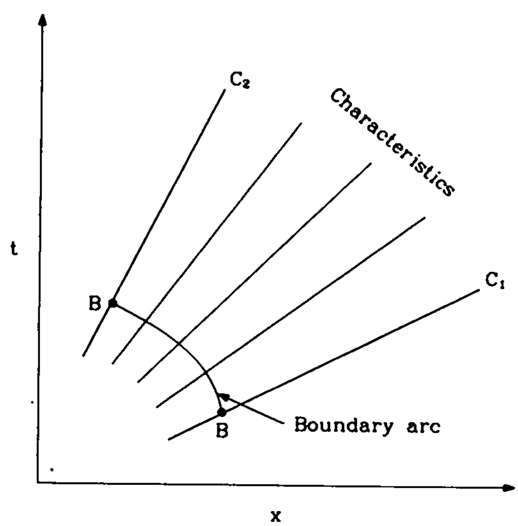


Fig. 2.1. Boundary data and characteristics.

be equal to the specified value of ρ at the point where it crosses the boundary arc. The specified value of ρ at each point on the boundary arc is thus propagated along the characteristic through that point at the characteristic speed $\bar{c}$. Notice that if we take for the boundary arc the unfortunate choice of a line segment in the characteristic direction, Fig. 2.2, the solution is determined only along the single characteristic line which is the extension of the boundary segment. Notice further that in this case we cannot make an arbitrary specification of the boundary data, for since the boundary arc lies on a characteristic the data along it must satisfy the characteristic equation ρ = constant. Thus we may specify any single value of ρ on the boundary arc but it must be constant everywhere on the arc.

From here on we shall stick to a special choice of boundary data, Fig. 2.3. We shall confine our attention to the first quadrant x = 0, t = 0, and take as the boundary arc the union of the positive x- and t-axes. Furthermore, we will ordinarily take a constant state on the x-axis (data on the x-axis is often called initial data), so that the interesting part of our boundary data is that on the t-axis. We shall call the boundary specification on the t-axis the piston. Applying it to the system generates a wave.

D. Problem 2.1. Solution of (2.1)

Given the boundary data

$$\rho = \rho_0 \neq 0 \quad \text{on} \quad t = 0 \quad , \quad \rho = \rho_b(t) \quad \text{on} \quad x = 0 \quad ,$$

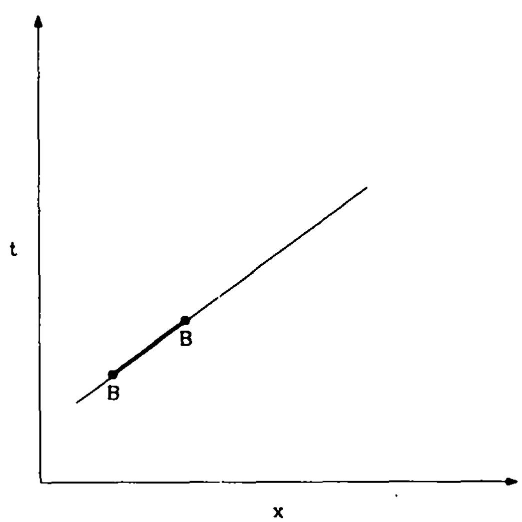


Fig. 2.2. An unfortunate choice of the boundary arc.

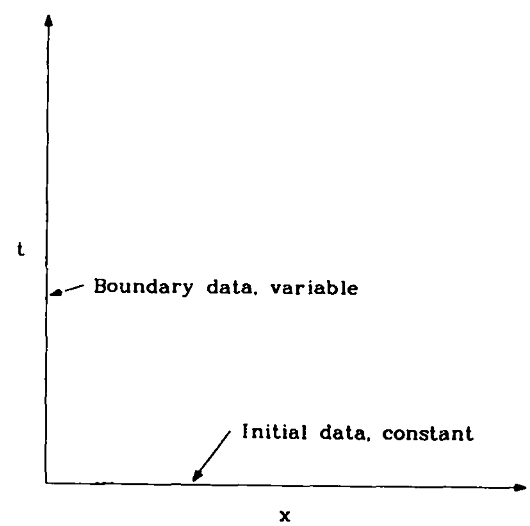


Fig. 2.3. Our standard boundary arc—the positive axes.

find the solution $\rho(x,t)$ of (2.1).

(1) Write the solution in parametric form $\rho(\tau)$ and $x = x(\tau,t)$, where the parameter τ is the time at which each characteristic intersects the t-axis, and

(2) Write the solution in explicit form $\rho(x,t)$ by eliminating τ.

(3) Choose (sketch) a function $\rho_b(t)$ and sketch the corresponding solution profile at some time.

E. Problem 2.2. Refraction

Let a wave of finite (spatial) extent pass from one material into another with different $\bar{c}$. How is the extent of the wave in each material related to the sound speeds?

F. Problem 2.3. Thermodynamics

Consider the differential equation for $T(p,v)$ obtained in problem 1.2.

1. Characteristic Form. Put the equation into characteristic form.

2. Identification. Identify the characteristic paths in the p-v plane with the level lines of a familiar thermodynamic function. Hint: use the first law and the differential of $e(p,v)$. Essentially this same identification was made in a problem of I.

3. γ-Law Gas. Specialize the results of (a) and (b) to the γ-law gas

$$e = pv/(\gamma - 1) \quad .$$

4. Ideal Gas. Take as boundary data T a linear function of p along the vertical line $v = v_0$, that is

$$T = (v_0/R)p \quad \text{on} \quad v = v_0$$

with R a constant. Find $T(p,v)$ everywhere.

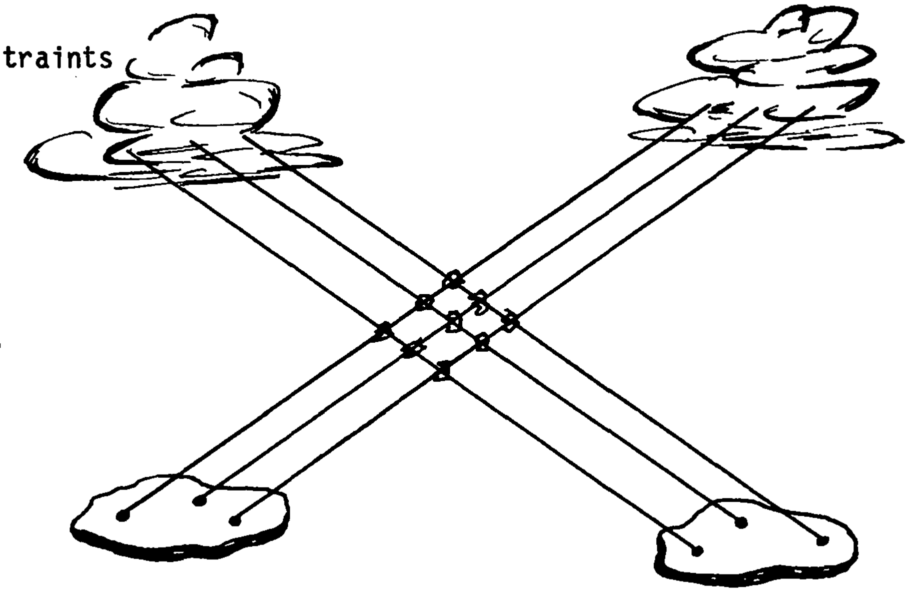
For a general approach to thermodynamics along these lines, see Cowperthwaite[1].

III. TWO LINEAR EQUATIONS

. . . many shall run to and fro,
and knowledge shall be increased.
-- Daniel 12:5

Keywords

boundary-data constraints
characteristics
interactions
p-u matching
reflection
reverberation
Riemann invariants
simple waves
spall
superposition
transmission



Adding a second dependent variable greatly complicates matters. However it will allow us to study a physically interesting system whose solutions include, in a fairly reasonable approximation, most of the one-dimensional applied problems of interest to us.

The main difference from the equations of Chapter 2 is that there are now two families of characteristics. This complicates the solution procedure somewhat, although all problems can still be solved exactly. The most important effect of the change is that there is now a much larger variety of solutions.

A. Differential Equation

The system we choose is again a simplified version of the general set. We simplify the equation of state by neglecting the dependence of p on e (that is, taking zero Gruneisen gamma), so the EOS $p(\rho,e)$ reduces to just $p(\rho)$. Thus the energy equation (1.1c) is not needed and we are left with (1.1a) and (1.1b) with independent variables p and u or ρ and u. We choose p and u, replacing the derivatives of ρ in (1.1a) by derivatives of p via the relation

$$dp = c^2 d\rho$$
$$c^2 = p'(\rho) \quad .$$

(Note that we are now using the usual physical definiation of sound speed). We next linearize the two equations about the constant state $p = \overline{p}$, $u = \overline{u} = 0$, adopting essentially the notation defined in (1.8)

$$\hat{p}(x,t) = \overline{p} + p(x,t)$$
$$\hat{u}(x,t) = u(x,t) \quad .$$

The resulting equations for the perturbations are

$$p_t + \overline{\rho c}^2 u_x = 0 \tag{3.1a}$$
$$u_t + \overline{v} p_x = 0 \tag{3.1b}$$
$$\overline{c}^2 \quad p'(\overline{\rho}) \quad ,$$

where, as before, the bar denotes the unperturbed state and the plain symbols p and u denote small perturbations on it.

B. Characteristics

The next step is to put these equations into characteristic form. There are several ways of doing this, each of which provides its own insights. We choose one of the simplest for presentation here.

To find the characteristic directions consider at each point (x,t) a linear combination of the two equations (3.1). We ask for those values of the combining coefficients such that, in the resulting linear combination, p and u are both differentiated in the same direction. This common direction is the characteristic direction.

Let the coefficients of the linear combination be 1 and b, with b to be determined. Taking 1 times (3.1a) plus b times (3.1b) gives

$$(p_t + b\overline{v}p_x) + (bu_t + \overline{\rho c}^2 u_x) = 0 \quad . \tag{3.2}$$

Now determine b by requiring that the ratio of the coefficients of p_x and p_t be the same as the ratio of the coeficients of u_x and u_t (that is, that in the linear combination (3.2) both p and u be differentiated in the same direction). This requirement is

$$\overline{\rho c}^2/b = b\overline{v}/1$$

or

$$b^2 = (\overline{\rho c})^2$$
$$b = \overline{\rho c} \quad .$$

There are two possible values of the combining coefficient b and thus two characteristic directions at each point (x,t). Substituting the first value $b = +\overline{\rho c}$ back into (3.2) we find

$$(p_t + \bar{c}p_x) + \overline{\rho c}(u_t + \bar{c}u_x) = 0 \quad . \tag{3.3}$$

We see that p and u are indeed differentiated in the same direction, and that this direction is

$$(dx/dt)_+ = \bar{c} \quad . \tag{3.4}$$

We distinguish the two characteristic directions by the subscripts (+) and (-). The subscript (+) here means "along a (+)-characteristic." Substituting (3.4) into (3.3) and recognizing the resulting directional derivatives we have

$$(dp/dt)_+ + \bar{z}(du/dt)_+ = 0$$
$$\bar{z} \equiv \overline{\rho c} \quad .$$

Repeating these steps with $b = -\bar{z}$ and collecting our results we find for the characteristic form of (3.1)

$$(dp/dt)_+ + \bar{z}(du/dt)_+ = 0 \quad \text{on} \quad (dx/dt)_+ = +\bar{c} \tag{3.5a}$$
$$(dp/dt)_- - \bar{z}(du/dt)_- = 0 \quad \text{on} \quad (dx/dt)_- = -\bar{c} \quad . \tag{3.5b}$$

These may be viewed as two ordinary differential equations, each holding on one family of characteristics. But note that they hold on different paths, so they are not the usual pair of coupled ordinary differential equations. Because the p-u equations in (3.5) have zero right side, we can eliminate dt to obtain

$$(dp/du)_+ = -\bar{z} \quad \text{on} \quad (dx/dt)_+ = +\bar{c} \tag{3.6a}$$
$$(dp/du)_- = +\bar{z} \quad \text{on} \quad (dx/dt)_- = -\bar{c} \tag{3.6b}$$

These can be integrated immediately to

$$R_+(p,u) \quad p + \bar{z}u = \text{constant} \quad \text{on} \quad x - \bar{c}t = \text{constant} \tag{3.7a}$$
$$R_-(p,u) \quad p - \bar{z}u = \text{constant} \quad \text{on} \quad x + \bar{c}t = \text{constant} \quad . \tag{3.7b}$$

The quantities R_+ and R_- are called Riemann invariants.

Now let us see how these are applied to the solution of a problem. Suppose we know the solution at two points 1 and 2 in (t-x) space, Fig. 3.1. We can find the solution at a certain new point 3 as follows. The location of the new point 3 in (t,x) space is the intersection of the (+)-characteristic through point 1

$$x - x_1 = \bar{c}(t - t_1) \quad , \tag{3.8}$$

and the (-)characteristic through point 2

$$x - x_2 = -\bar{c}(t - t_2) \quad . \tag{3.9}$$

Along the plus characteristic we have

$$p - p_1 = -\bar{z}(u - u_1) \quad , \tag{3.10}$$

and along the minus characteristic we have

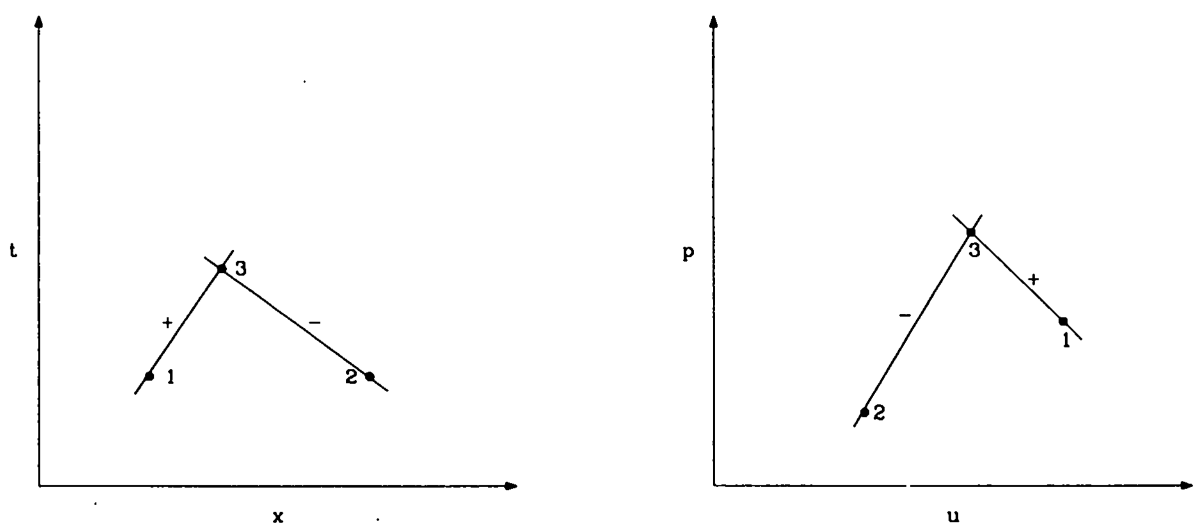


Fig. 3.1. Finding the solution at point 3 from that at points 1 and 2.

$$p - p_2 = \bar{z}(u - u_2) \quad . \tag{3.11}$$

Just as both (3.8) and (3.9) must be satisfied simultaneously at point 3 in (t,x) space, so must (3.10) and (3.11) be satisfied simultaneously at point 3 in p-u space; the (p-u) intersection gives the solution (p_3,u_3) at (t_3, x_3). We now have the complete state at point 3. The process of finding the state at the new point is thus much like the standard p-u matching at interfaces.

In general, our entire solution is built up in this way. We begin with all pairs of points (like 1 and 2 above) lying on the boundary consisting of the positive x- and t-axes. Now on the x-axis we can specify any values of p and u at each point. Having done this, we are restricted in how much we can specify on the t-axis: we can specify p(t) or u(t), but not both. This comes about because each point on the t-axis is intersected by a (-)-characteristic coming up from the x-axis, as shown in Fig. 3.2. Because this (-)-characteristic carries a relation between p and u which must be satisfied, specifying either one where it intersects the x-axis determines the other. Thus specifying either $u_b(t)$ or $\rho_b(t)$ (on the t-axis) determines the other, and the values of p and u are known on the entire boundary.

C. Simple Waves

We distinguish two regions of t-x space (other than a constant state): (1) simple waves, and (2) interactions. For linear equations both can be

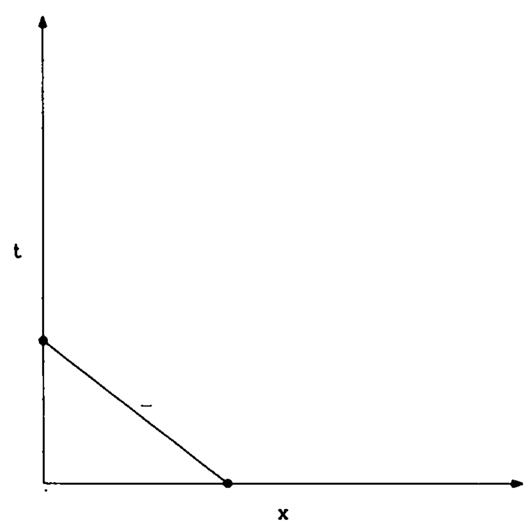


Fig. 3.2. Either p or u, but not both, may be specified on the t-axis.

calculated exactly, but the simple wave is the simpler. Although the linear case is somewhat degenerate, we define the simple wave here and discuss it briefly because it results in a more important simplification when we come to the nonlinear case.

A simple wave is defined as any region of the flow adjacent to a constant state, Fig. 3.3, the dividing line bounding the constant-state region being a characteristic. Its most important property is that in p-u space it lies entirely on one curve, the characteristic through the constant state adjoining the simple wave. This is readily seen to be the case. Consider all (-)-characteristics entering the simple wave from the constant state, such as those shown in the figure. These cover the t-x space of the simple wave and each starts in the constant state, so the entire simple wave maps into this one (-)-characteristic in p-u. Furthermore, the state along any (+)-characteristic in the simple wave is constant. To see this, consider the (+)-characteristic through the state 2 in the figure. Let its state (which must lie on the (-)-characteristic through state 1) be as shown in the p-u plane. Now calculate any other state, say 2', on it in the standard way, by intersecting, in p-u, the (+)-characteristic through 2 with the (-)-characteristic through 1. The result, as seen from the figure, is that 2' is the same as 2; hence the state is the same everywhere on the (+)-characteristic.

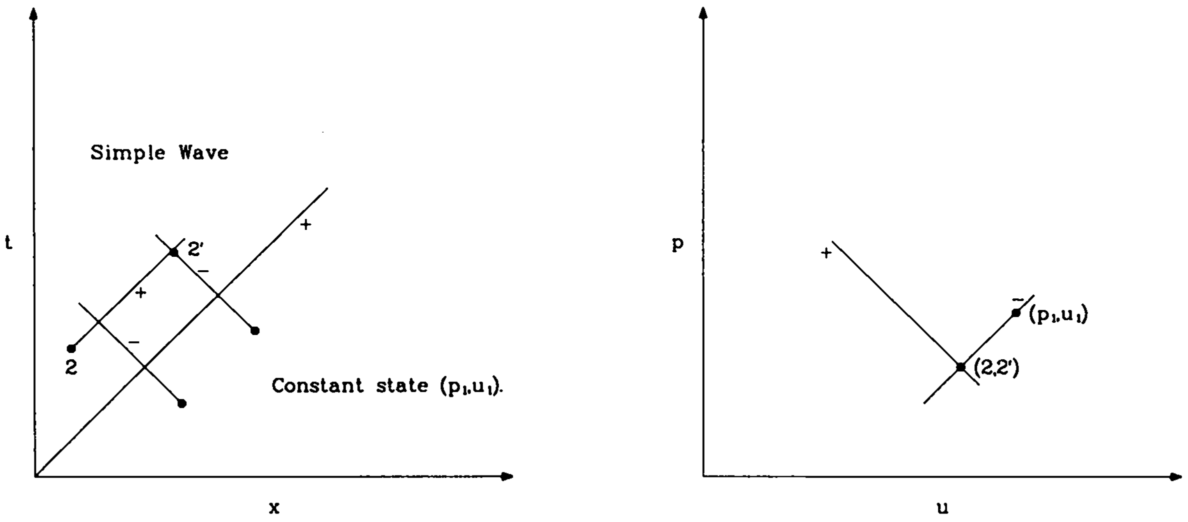


Fig. 3.3. The simple wave.

In contrast to a simple wave, an interaction region is adjacent to a non-constant state, and its map in p-u space occupies a finite area. This will be amply illustrated in the problems, and we shall not discuss it further here.

Another way of distinguishing simple waves and interactions is by considering superposition of waves. A simple wave is a region having only a single wave. An interaction is a sum of two waves running in opposite directions.

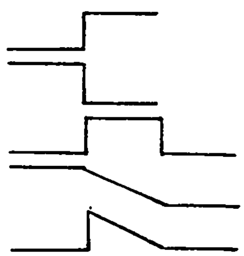
D. Problems

The following remarks apply to the problems other than problem 3.1.

1. Standard Material. If not otherwise specified, the material is a solid with infinite tensile strength, initial pressure $p_0 = 0$, and initial velocity $u_0 = 0$. (Without going into detail, we state that our equations describe longitudinal waves in a linear elastic solid.)

2. Standard Waveshapes. We name these as follows. Each is generated by the u vs t (piston) function sketched, having unit amplitude and (for the last three) unit duration.

(1) shock
(2) jump rarefaction
(3) square wave
(4) ramp rarefaction
(5) "detonation"



3. Displays. Present some selection from the following sketches: t-x diagram, p-u diagram, snapshots (p-x and u-x) at several times, and particle histories. Where it is reasonable to do so, keep the particles at their original positions the first time around. Particle paths can then be sketched and the results modified if necessary.

E. Problem 3.0. ρ-u Riemann Relation.

Find the Riemann relation between ρ and u.

F. Problem 3.1. Simple Wave = Kinematic Wave

Show that the flow within a simple wave obeys a kinematic wave equation by deriving a kinematic wave equation for it in three different ways. (Hint: the constancy of the Riemann function furnishes the kinematic "EOS".)

(1a) Use the result of problem 3.0 in the linearized mass conservation relation $\rho_t + \bar{\rho} u_x = 0$.

(1b) Use the p-u Riemann relation (3.7b) in (3.1a).

(2) Use the p-u Riemann relation (3.7b) in (3.1b).

G. Problem 3.2. Wave Generation

(1) Sketch the wave shape generated by each standard piston motion given above.

(2) How does the wavelength depend on $\bar{c}$?

(3) How does the pressure amplitude depend on $\bar{z}$?

(4) If we had specified piston pressures instead of velocities (of the same shape and unit pressure amplitude) how would the velocity amplitude depend on $\bar{z}$?

H. Problem 3.3. Reflection

Let each of the standard waves reflect from

(1) a rigid wall ($u = 0$)

(2) a free surface ($p = 0$).

Note that free-surface reflection _________ the wave. Note that when a shock reflects from a free-surface, the free-surface velocity is _________ the original particle velocity behind the shock.

I. Problem 3.4. Spall

Let a detonation wave run into a free surface, as in problem 3.2, but this time let the material have finite strength. To simplify the analysis, assign a tensile strength of one-half the peak wave amplitude to only a single particle located far enough from the free surface to be outside the interaction region, and assume that it spalls (breaks) a short but finite time after first experiencing tension.

Spalling can be avoided by moving the single breakable particle close enough to the free surface (inside the interaction region). How close to the surface must it be so as not to spall?

J. Problem 3.5. Ringing

Consider a plate of finite thickness with a free-surface ($p = 0$) at its right boundary. At its left boundary apply the following boundary condition to generate a square wave

$$p = 0 \quad \text{for} \quad t < 0$$
$$p = 1 \quad \text{for} \quad 0 \le t \le 1$$
$$p = 0 \quad \text{for} \quad t > 1.$$

Take the plate thickness several times the wavelength. Follow the motion through two reflections. What is the time-average velocity of the plate?

K. Problem 3.6. Gas Gun

A semi-infinite tube of gas at pressure p_0 is closed at one end by an incompressible piston of mass m per unit area with p = 0 outside the piston. The piston is clamped in place until t = 0 and then released. Write and solve the ordinary differential equation for the piston motion u(t).

Consider also the case in which the target consists of three pieces (in contact), each the same thickness as the projectile. Recall the analogous experiment with steel balls.

L. Problem 3.7. Colliding Plates

A projectile plate moving at velocity u_0 strikes a stationary plate of the same material. Consider the three cases in which the thickness of the projectile plate is greater than, less than, or equal to that of the target.

M. Problem 3.8. Reflection/Transmission

Let each of the standard waves pass into an adjacent material of (a) higher, and (b) lower impedance. Take both pieces semi-infinite and the interface glued with infinite strength. For a continuous incoming wave like the ramp rarefaction, how does the relative steepness of the transmitted and incident waves depend (qualitatively) on the impedance ratio?

Note that the entire flow in the first material may be calculated by replacing the second material by a right boundary condition in which neither p nor u alone is specified, but which consists of a linear relation between them. What is this linear relation?

N. Problem 3.9. Driven Plate (Reverberation)

Let a shock pass from semi-infinite material A into a finite-thickness plate of material B. Consider four cases, the possible combinations of

(a) $\bar{z}_B > \bar{z}_A$ and $\bar{z}_B < \bar{z}_A$

and (b) interface glued (infinite tensile strength) and not glued (zero tensile strength).

For which impedance ratios do the plates want to separate and how strong must the glue be to hold them together?

For the unglued, no-separation case, make a qualitative sketch of the free-surface velocity vs time.

Pullback: discuss qualitatively how this velocity history would change if the incoming wave were the detonation instead of the shock.

O. Problem 3.10. Flyer

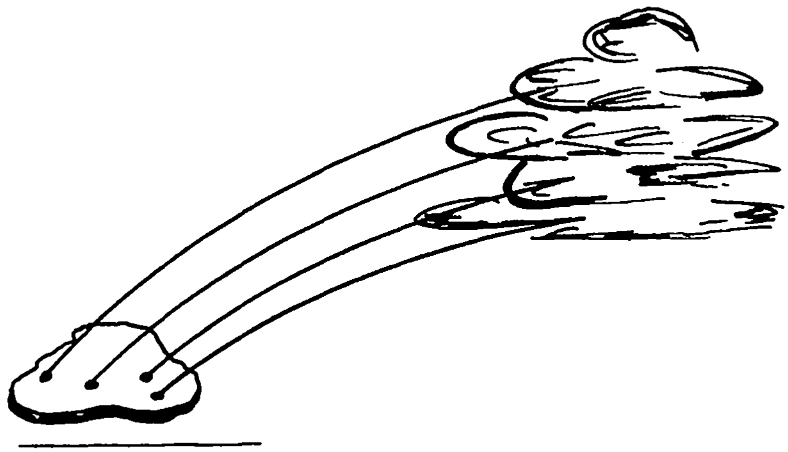
A finite-thickness plate A (the flyer) moving with velocity u_0 impacts a semi-infinite plate B (the target).

For what impedance ratios will the flyer bounce off the target?

IV. ONE NONLINEAR EQUATION

Wie Schlänglein krumb
Gehn lächelnd umb
Die Bächlein kühl in Wälden.
-- Friedrich Spee, Liebgesang

Keywords
amplitude dispersion
centered rarefaction wave
Hugoniot
nonlinearity
Rayleigh line
self-similar flow
shocks and shock formation



We now introduce nonlinearity. We retreat to one equation in one unknown, as in Chapter 2, but now take the complete equation of state $p(\rho)$ instead of linearizing about one point. The equation thus has a coefficient which is a function of the dependent variable ρ.

As in Chapter 2, we have just one family of characteristics, but they now depend on the given boundary conditions, instead of being fixed ahead of time. They have in general different slopes, giving rise to the phenomenon of amplitude dispersion.

A. Differential Equations

We introduce nonlinearity by taking the system (1.6)

$$\rho_t + p_x = 0 \tag{1.6}$$

$$\text{or} \quad \rho_t + c(\rho)\rho_x = 0$$

$$c = p'(\rho) \quad .$$

It suffices here to take

$$p = 1/2\ \rho^2$$
$$c = \rho\ ,$$

giving

$$\rho_t + \rho\rho_x = 0\ , \tag{4.1}$$

although any $p(\rho)$ with $p'(\rho) > 0$ and $p''(\rho) > 0$ would give similar results.

B. Characteristics

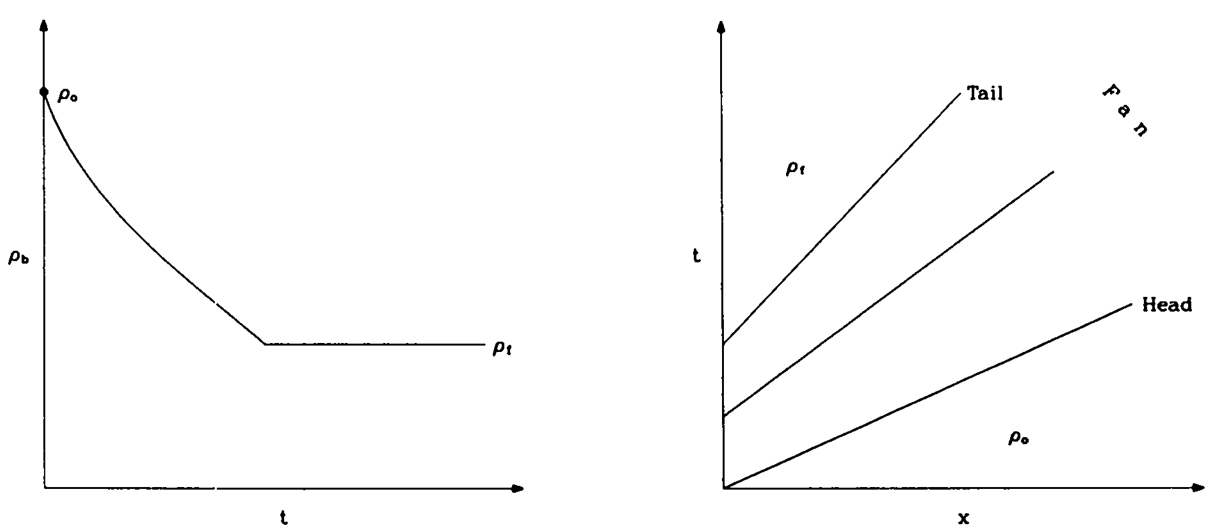
The characteristic form of (4.1) is

$$d\rho/dt = 0 \quad \text{on} \quad dx/dt = \rho\ . \tag{4.2}$$

Hence the characteristics are still straight lines, as in Chapter 2, but they may now have varying slopes. This change is quite important, as we shall see.

C. Rarefaction

For monotone decreasing boundary density $\rho_b(t)$ we have a rarefaction wave, Fig. 4.1. As before each boundary value is propagated along its characteristic, but since ρ_b is decreasing with time the characteristics now fan out as shown, and the wave exhibits amplitude dispersion or spreading. As indicated in the figure, because of its shape in t-x, the wave is sometimes



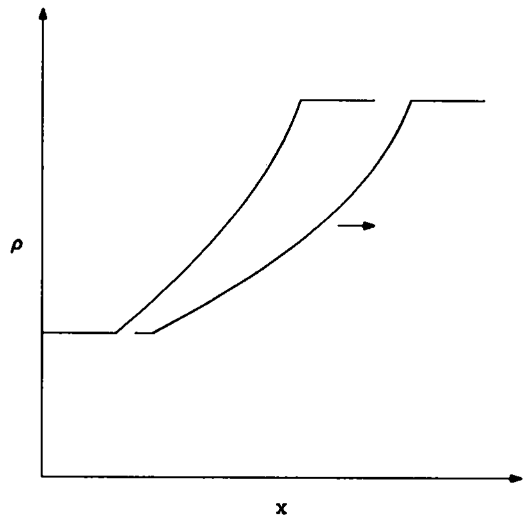


Fig. 4.1. The rarefaction wave.

called a fan. The head and tail are the characteristics bounding the fan on the right and left.

The solution in parametric form (Problem 2.1) is

$$\rho = \rho_b(\tau) \quad \text{on} \quad x = \rho_b(\tau)(t - \tau) \quad .$$

The parameter τ labels the characteristics; its value for each characteristic is the time at which that characteristic intersects the x-axis. For sufficiently simple $\rho_b(\tau)$ we can eliminate τ and write an explicit expression for $\rho(x,t)$. For example, let $\rho_b(t)$ be the ramp function

$$\begin{aligned} \rho_b &= \rho_0 && \text{for} \quad t < 0 \\ \rho_b &= \rho_0(1 - t) && \text{for} \quad 0 \le t \le 1/2 \\ \rho_b &= \rho_0 && \text{for} \quad t > 1/2. \end{aligned}$$

Within the fan we have

$$\rho = \rho_0(1 - \tau) \quad \text{on} \quad x = \rho_0(1 - \tau)(t - \tau) \quad .$$

Eliminating τ and solving for $\rho(x,t)$ gives the quadratic equation

$$\rho^2/\rho_0 + (t - 1)\rho - x = 0$$

for $\rho(x,t)$.

Another way of solving this problem is to note that the general solution of (4.1) is

$$\rho = f(\xi) \quad , \; \xi = x/\rho - t \quad ,$$

with f an arbitrary function. Evaluating this on the boundary x = 0 determines f, giving the solution immediately in implicit form

$$\rho = \rho_b(x/\rho - t) \quad .$$

A common and particularly simple configuration is the centered rarefaction wave, Fig. 4.2, the limit of the wave of Fig. 4.1 as the time over which ρ_b drops to its final velocity approaches zero. All of the characteristics of the fan now emanate from the origin. The solution in the fan is

$$\rho = x/t \quad ,$$

as can be verified by substitution. A flow such as this which is a function of a single variable which is some combination of x and t is called self-similar.

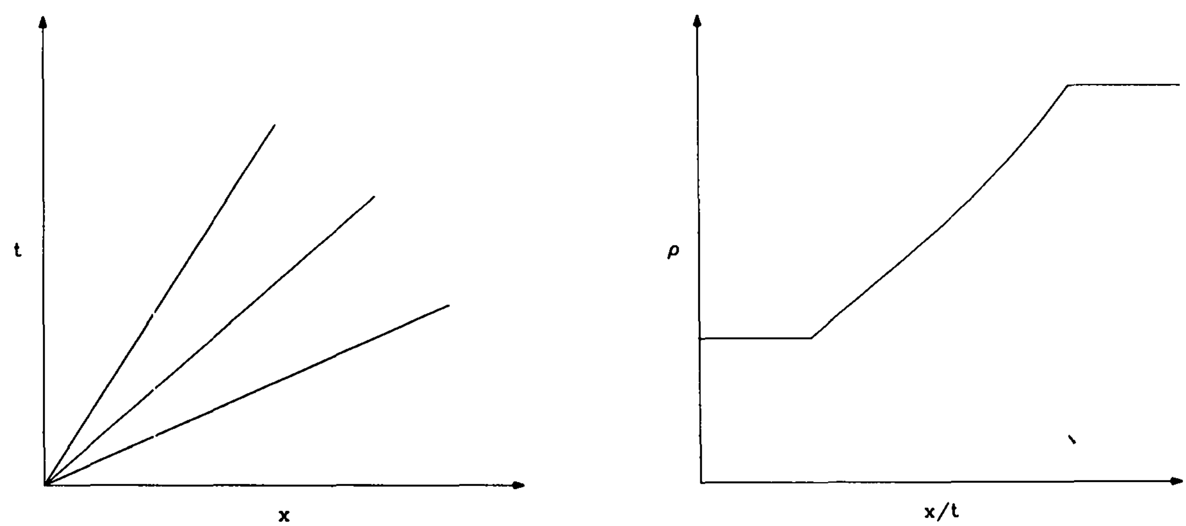


Fig. 4.2. The centered rarefaction wave.

D.... Compression

With ρ_b increasing with time we have a compression wave, Fig. 4.3. Successive characteristics now converge instead of fanning out and will eventually cross, leading to a nominally triple-valued solution.

Now the usual physical requirement on the solution is that ρ be single-valued, and we replace the triple-valued solution by a moving jump discontinuity, or shock. The shock forms with zero strength at the point of first crossing of the characteristics and then grows in strength as it overtakes characteristics ahead of itself and is overtaken by characteristics from behind.

E. Shock

The shock velocity D depends on the states on either side of the shock. The easiest way to derive this shock-jump or Hugoniot relation is to appeal to the physics. Denoting the state ahead of the shock by subscript zero and that behind by a plain symbol, we equate the mass flux u on each side

$$\rho_0(u_0 - D) = \rho(u - D)$$

or, using the definition (Chapter 1) $p = \rho u$

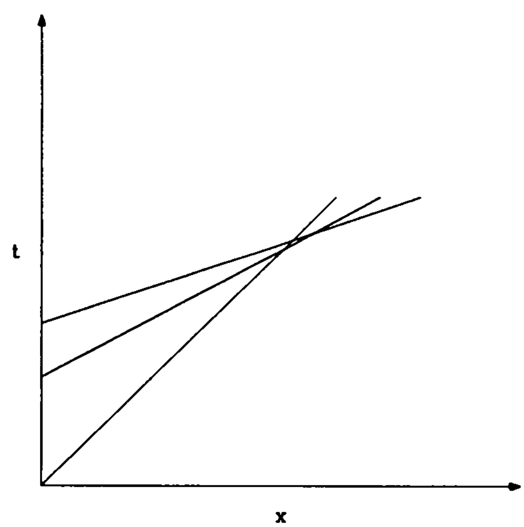


Fig. 4.3. Compression wave with shock formation.

$$D = \frac{p - p_0}{\rho - \rho_0} \quad .$$

In the p-ρ plane this is a straight line of slope D through the initial state, Fig. 4.4. The upper point S at which the Rayleigh line crosses the EOS p(ρ) is the shocked state for this velocity. For our EOS $p = 1/2\ \rho^2$ the jump relation becomes

$$D = 1/2\ (\rho + \rho_0) = 1/2\ (c + c_0) \quad ,$$

that is, the shock speed is the mean of the characteristic speeds before and behind. This is a special case of the general property enunciated earlier: a shock overtakes characteristics ahead and is overtaken by characteristics from behind, Fig. 4.5.

The compression analog of the centered rarefaction wave, that is, the response to a step-function $\rho_b(t)$, is a flat-topped shock.

F. Problems

Take (4.1) as the equation of motion, with $\rho_0 = 1$ throughout.

1. Problem 4.1. Shock Degradation by Rarefaction. Take a square-wave boundary function which generates first a shock and then a centered rarefaction

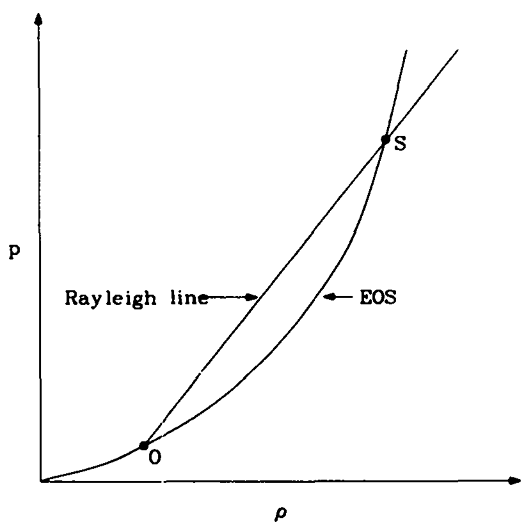


Fig. 4.4. The Rayleigh line.

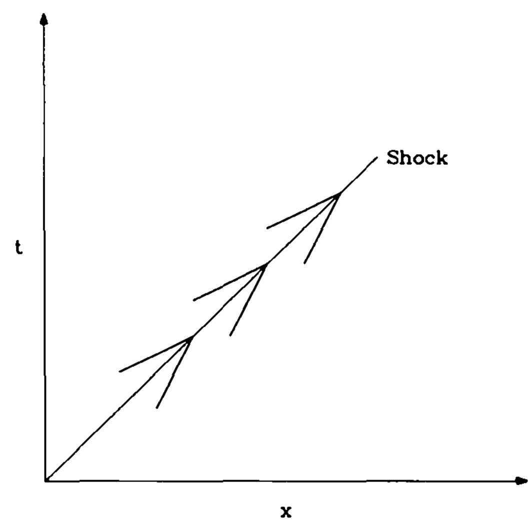


Fig. 4.5. Shock overtaking characteristics ahead and being overtaken by characteristics from behind.

$$\rho_b = 1 \quad \text{for} \quad t < 0$$
$$\rho_b = 3 \quad \text{for} \quad 0 \le t \le 1$$
$$\rho_b = 1 \quad \text{for} \quad t > 1 \quad .$$

Find the point at which the head of the rarefaction overtakes the shock. Then write and solve the ordinary differential equation for the subsequent shock path. Hint: good new variables are

$$\hat{t} = t - 1 \quad , \quad y = x_s(t)/\hat{t}$$

where $x_s(t)$ is the shock path.

How long does the overtake last? What is the late-time dependence of shock strength on t?

<u>2. Problem 4.2. Shock Formation.</u> Find the point of shock formation (earliest crossing of characteristics) for the two boundary conditions

(a) linear piston

$$\rho_b(t) = 1 \quad \text{for} \quad t < 0$$
$$\rho_b(t) = 1 + at \quad \text{for} \quad 0 \le t \le 1$$
$$\rho_b(t) = 1 + a \quad \text{for} \quad t > 1$$

(b) quadratic piston

$$\rho_b(t) = 1 \quad \text{for} \quad t < 0$$
$$\rho_b(t) = 1 + at^2 \quad \text{for} \quad 0 \le t \le 1$$
$$_b(t) = 1 + a \quad \text{for} \quad t > 1 \quad .$$

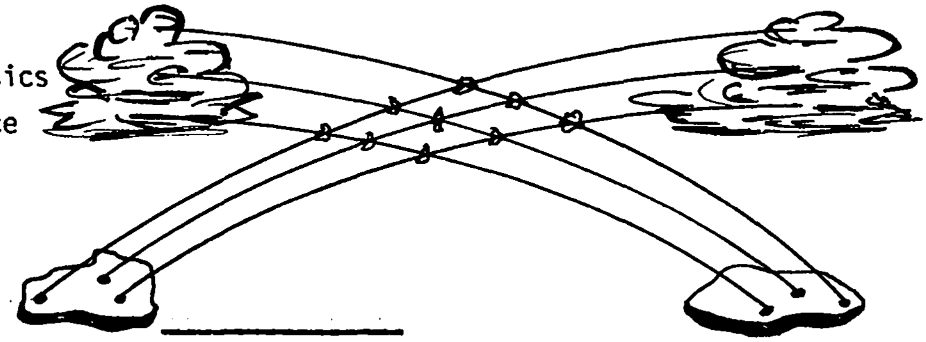
Sketch a few snapshots before and after shock formation for each.

V. TWO NONLINEAR EQUATIONS

Wie Ast an Ast sich ächzend reibt und knackt,
Wie Blitz an Blitz durch Schwefelgassen zuckt.
-- Annette von Droste-Hülshoff,
Am dritten Sonntag nach Ostern

Keywords

curved characteristics
domain of dependence
interactions
range of influence
simple waves



The next step is like that between Chapters 2 and 3. We add the momentum equation to get a system of two equations in two unknowns. The equation of state $p(\rho)$ is unchanged. In addition to the effects of adding a second equation in the linear approximation, we have the important result that the characteristics in the interaction regions are curved. This comes about because of the nonlinearity of the equations with the concomitant dependence of the characteristic slope on the state. Because of this curvature and state dependence, the solution at any one point now depends not on just two points of the boundary, but on the entire boundary arc cut off by the (+) and (-) characteristics through the point in question.

A. Differential Equations

We consider the same system which we linearized to get the equations for Chapter 3: the mass and momentum equations (1.1a) and (1.1b) plus the restricted equation of state $p(\rho)$

$$\rho_t + u\rho_x + \rho u_x = 0 \qquad (5.1a)$$

$$u_t + uu_x + vp_x = 0 \qquad (5.1b)$$

$$p = p(\rho) \quad . \qquad (5.1c)$$

B. Characteristics

Analysis like that of Chapter 3 gives for the characteristic equations (exercise for the student)

$$(dp/du)_+ = -z(p) \quad \text{on} \quad (dx/dt)_+ = u + c \qquad (5.2a)$$

$$(dp/du)_- = +z(p) \quad \text{on} \quad (dx/dt)_- = u - c \quad , \qquad (5.2b)$$

where we have taken, as before, p and u as independent variables. Here $c = p'(\rho)$ can be expressed as a function of p alone as can $z = \rho c$, since we have $\rho = \rho(p)$ by inverting the equation of state.

Things are now much more complicated than in the linear case. The location of the characteristics in t-x space can not be determined in advance, for their (changing) slope depends on the dependent variables and thus on the given boundary data. The p-u equations can, fortunately, be solved, but their solution is more complicated because z is a function of p instead of a constant. The solution is

$$R_+(p,u) = u + \int dp/z(p) = \text{constant}$$
$$R_-(p,u) = u - \int dp/z(p) = \text{constant} \quad .$$

As before, the functions R_+ and R_- are called Riemann invariants. Given the equation of state and thus z(p), these functions can be calculated and the characteristic paths in p-u space determined (without knowing anything about the characteristics in t-x).

Consider now the process of finding the solution at a new point given its values at two previous points, as in Fig. 3.1. The p-u state at the new point is known, but its location in t-x space is not -- we know what the new state is but don't know where it is.

To get a feeling for the Riemann function we write it down for a γ-law gas

$$p = k\rho^\gamma \quad , \quad \gamma \text{ constant} \quad . \qquad (5.3)$$

Taking k = 1 for simplicity, we have (exercise for the student)

$$\begin{aligned} c^2 &= \gamma p/\rho = \gamma p^{(\gamma-1)/\gamma} \\ z &= \rho c = \gamma^{1/2} p^{(\gamma+1)/2\gamma} \\ R_+ &= u \pm [2/(\gamma - 1)]\gamma^{1/2} p^{(\gamma-1)/2\gamma} \\ &= u \pm [2/(\gamma - 1)]c \quad . \end{aligned} \qquad (5.4)$$

Taking $\gamma = 3$ results in a great simplification.

$$c = \sqrt{3p}^{1/3}$$
$$R_\pm = u \pm \sqrt{3p}^{1/3}$$
$$= u \pm c$$

For this special case u + c is constant along the (+)-characteristics and u - c is constant along the (-)-characteristics so both families are straight lines in t-x (since their slopes are just $u \pm c$). Solving the nonlinear problem is not much harder than solving the linear one. The characteristics in p-u space are curved, but even this slight difficulty can be removed (for a single material) by working in c-u space instead.

To simplify our discussions, we shall choose a boundary condition which is not the usual physical one, namely, values of $u_b(t)$ on x = 0. (For the usual solid piston, the piston position is $x = \int u_b(t)dt$.). Physically, specifying u on x = 0 corresponds to a porous piston through which some matter flows.

C. Rarefaction and Simple Waves

The distinction between simple waves and interaction regions in Chapter 3 applies here also, with a greater difference in complexity here. Within a simple wave the governing partial differential equations again reduce to an ordinary differential equation.

A simple wave is defined as a region of the flow adjacent to a region of constant state, and containing no jump discontinuities (shocks). It has two nice properties: (1) the entire wave lies on a single characteristic curve in p-u space, and (2) each characteristic of the family facing in the same direction as the wave is a straight line in t-x, and the state along it is constant.

We demonstrate these properties for a simple rarefaction wave, with some repitition of the argument of Chapter 3.

Let the simple wave be a right-facing rarefaction wave moving into a constant state (p_0, u_0), generated by a specified monotone decreasing $u = u_b(t)$ on x = 0 , t > 0, Fig. 5.1. Note that in t-x the (-)-characteristics all start in the constant-state region, and that the (+)-characteristics all start on x = 0. The proof proceeds in three steps:

(1) The entire simple wave lies on the (-)-characteristic through (p_0, u_0) in p-u space. This follows immediately from the t-x diagram and the assumptions: Consider any (-)-characteristic in t-x. It starts in the initial state (p_0, u_0). With no shocks, the

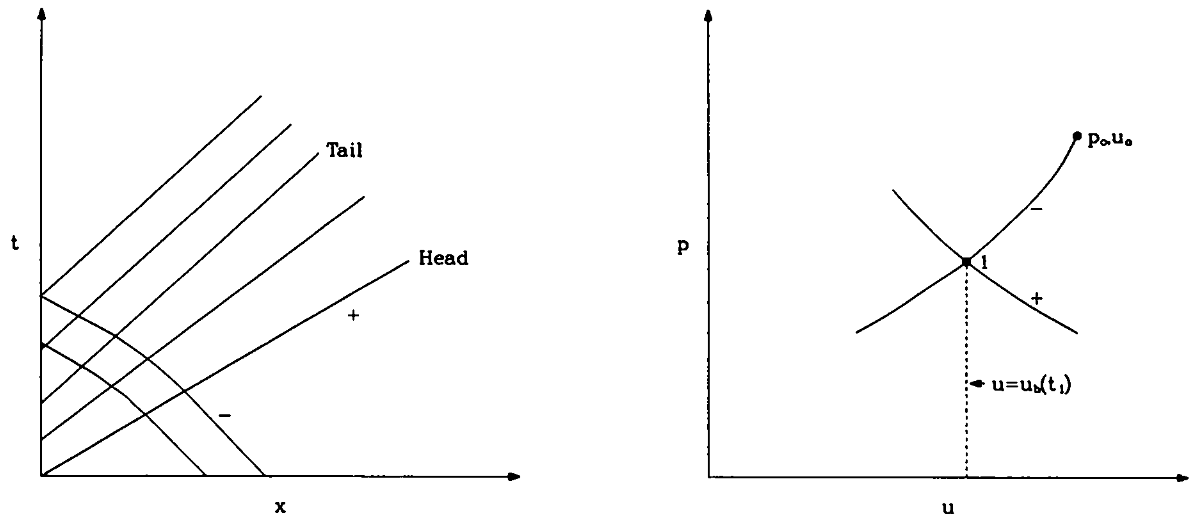


Fig. 5.1. The simple wave.

differential equations apply everywhere on it, that is $R_-(p,u) = R_-(p_0,u_0)$, so that all states on it lie on the (-)-characteristic through (p_0,u_0) in (p-u). Since this is true for all (-)-characteristics, it is true for the entire flow.

(2) Every (+)-characteristic in t-x is a straight line and the state is constant along it. Proof: Since the entire wave lies on the (-)-characteristic in p-u, so do the boundary points on the t-axis. Consider one of these, point 1 in the figure (Its pressure p_1 is the intersection of the vertical line $u = u_b(t_1)$ with the (-)-characteristic in p-u.) Now consider any point on the (+)-characteristic through point 1. Its state is the intersection of the (+)- and (-) p-u characteristics through state 1, that is, just state 1 itself, as is evident from the figure. Hence the state is constant everywhere on this (+)-characteristic and it is therefore a straight line in t-x by the t-x equation of (5.2a).

As in Chapter 4, the simplest rarefaction wave is a centered rarefaction wave. For any EOS the solution is

$$x/t = u + c \quad ,$$

with the p-u relation through the wave depending on the particular EOS.

D. Compression Waves

A simple compression wave can of course be generated by taking an increasing $u_b(t)$ and constant initial state. Such a wave remains strictly simple only up to the time of shock formation, but for the simple system considered here, the motion beyond that time is easily calculated. The process of shock formation is essentially the same as in the system of Chapter 4 (Problem 4.2).

E. Shocks

As we saw in I, the mass and momentum equations lead to the Rayleigh-line relation (independent of the equation of state)

$$\rho_0^2 D^2 = \frac{p - p_0}{v_0 - v} ,$$

a straight line in p-v slope of $\rho_0^2 D^2$. With our restricted equation of state p(v), we have no separate Hugoniot curve; all states whatsoever lie on the EOS p(v).

Thus all shock states lie on p(v). For a shock of velocity D, the shock state in p-v is just the intersection of the Rayleigh line for this D with the state curve p(v), Fig. 5.2. The picture in the p-u plane is topologically the same. The Rayleigh line there is given by

$$(p - p_0) = (\rho_0 D)u ,$$

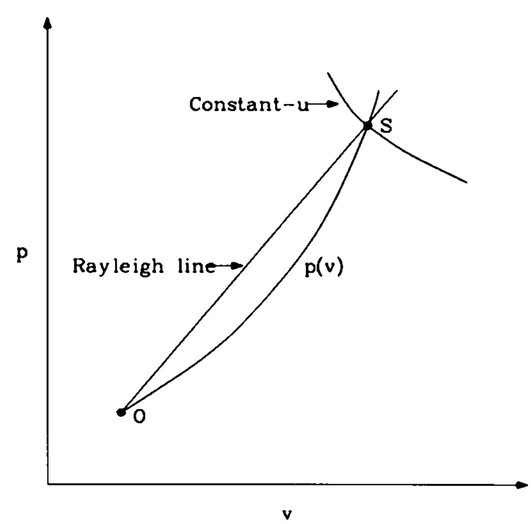


Fig. 5.2. The shock state S as the intersection of the Rayleigh line and the state curve.

and the analog of the state curve is

$$u^2 = (p - p_0)(v_0 - v(p)) \quad ,$$

with v(p) the (inverted) EOS. We may also consider the loci of constant u in the p-v plane

$$u^2 = (p(v) - p_0)(v_0 - v) \quad .$$

These are a family of hyperbolae, with u as the parameter, Fig. 5.2. Note that specifying either D or u determines the shock state (for given initial state).

It is easily shown (qualitative exercise for the student) that a shock bisects the characteristic field in the same way as for the single nonlinear equation, Fig. 4.5.

F. Problems

1. Problem 5.1. Simple Wave = Kinematic Wave. The flow in a simple wave satisfies a kinematic wave equation. As in problem 3.1, the relation R_- = constant provides the desired "EOS." Write this kinematic wave equation with p as the independent variable, starting from (5.1b).

2. Problem 5.2. Characteristic Curvature. Consider the collision of two simple rarefaction waves in a γ-law gas. In t-x, the (+)-characteristics of the wave coming from the left and the (-)-characteristics of the wave coming from the right are straight lines before the collision. Which way do they curve within the interaction region for (a) $\gamma < 3$, (b) $\gamma > 3$? Answer the same question for two simple compression waves (assume no shocks form).

3. Problem 5.3. Simple-Wave Paths. Write the ordinary differential equations for the following paths through a forward-facing centered rarefaction wave: (a) particle path, and (b) (-)-characteristic. Use y = x/t as the independent variable.

Solve the particle-path equation for the γ-law EOS $p = \rho^\gamma$.

4. Problem 5.4. Plate Push. A semi-infinite tube of a γ-law gas is bounded on the right by an incompressible piston of mass m per unit area, clamped in place until time t = 0 and then released. Write and solve the differential equation of the piston motion.

5. Problem 5.5. Wave Reflection and Refraction. Discuss qualitatively the impingement of (a) a flat-topped shock, and (b) a narrow-fan rarefaction wave, on (1) a rigid wall, (2) a free surface, and (3) a second material of lower and of higher impedance.

What are the main differences from the linear case?

6. Problem 5.6. Wave Interactions. Consider qualitatively all cases of collision (of two waves facing in opposite directions), and overtake or lack of it (of two waves facing in the same direction) for all combinations of the following two waves: (1) a flat-topped shock, and (2) a weak (narrow-fan) centered rarefaction. Consider the Tait isentrope

$$p = a\ [(\rho/\rho_0)^\gamma - 1]$$

or, for $\rho_0 = 1$,

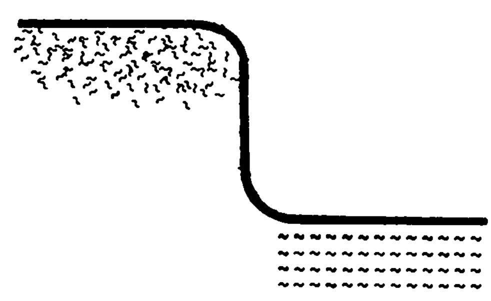
$$p + a = a\rho^\gamma\ .$$

For $\rho_0 = 1$, $p_0 = 0$, $a = 1$, and $\gamma = 3$ calculate p and u for (a) a shock, and (b) a compression wave, at $\rho = 1.2$ and 1.4.

VI. THE FULL (THREE) EQUATIONS

Is crumbled out to his Atomis.
'Tis all in pieces, all coherance gone. . .
-- John Donne, The First Anniversarie

Keywords
entropy production
Hugoniot
isentropes
particle paths



We have finally arrived at the full equations (1). The required changes in the equations (5.1) of the preceding chapter are those needed to take account of the energy: the addition of the energy conservation equation (1.1c), and the extension of the EOS from p(v) to p(v,e), so that it depends on both volume and energy.

The main new feature is the entropy production by the shock. For conceptual purposes it is useful to think of the EOS in the form p(v,s), with the isentropes (curves of constant entropy s) of particular interest. A particle's entropy remains constant, that is, it stays on the same isentrope in p-v, until it passes through a shock, which increases its entropy and displaces it to a higher isentrope. Thus, for example, a particle which is shocked and then rarefied (isentropically expanded) back to its original pressure ends up at a higher temperature -- the result of its increased entropy induced by the trauma of its passage through the shock.

What are the consequences of this entropy production by the shock? The first is that we have a new family of thermodynamic loci, the Hugoniot curves. Each of these is the locus of all possible shock states for a given initial state. Now the equation of state p(v,s) may be regarded as a one-parameter family of isentropes with parameter s. A Hugoniot curve in p-v is steeper than the isentropes everywhere except at its initial state, as seen in Fig. 6.1. Also, in contrast to the isentropes, which approach infinite volume at zero pressure, a Hugoniot has a vertical asymptote at finite volume.

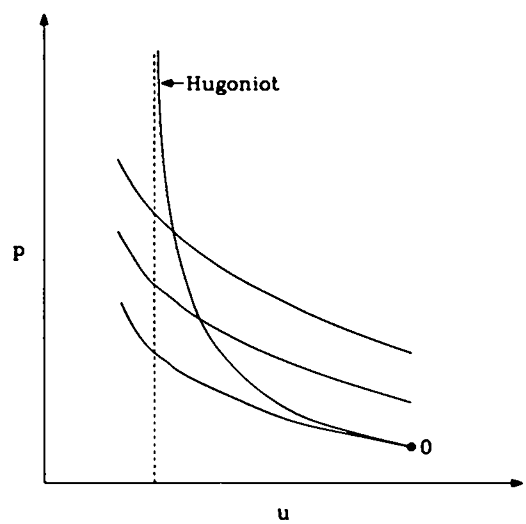


Fig. 6.1. Hugoniot and isentropes.

The second consequence of entropy production in the shock is that a shock of varying strength produces a more complicated flow field -- one of varying entropy. Each particle passing thru such a shock experiences a jump of different strength, so that the shocked region has an entropy gradient. Problems like the degradation of a shock by an initially simple rarefaction wave are no longer soluble, except for especially chosen equations of state (see, for example, Fickett[2].

A. Differential Equation

Our system is the set of equations (1)

$$\dot{\rho} + \rho u_x = 0 \tag{1.1a}$$

$$\dot{u} + v p_x = 0 \tag{1.1b}$$

$$\dot{e} + \dot{p} v = 0 \tag{1.1c}$$

One route to the characteristic equations is to eliminate derivatives of the inverted EOS e(p,v)

$$de = e_p dp + e_v dv \quad . \tag{6.1}$$

Putting this in (1.1c) we obtain

$$\dot{p} = c^2 \dot{\rho} \quad , \tag{6.2}$$

where c^2 is a symbol for

$$c^2 = -v^2(p + e_v)/e_p \quad . \tag{6.3}$$

Before proceeding, we make a few remarks about the significance of the quantity c. If we are willing to make use of classical thermodynamics, we can identify the right side of (6.3) with $(\partial p/\partial \rho)_s$, as shown in problem 2.3. Thus (6.2) states that the entropy of a particle remains constant (in any continuous region of the flow). Another approach is to minimize the number of assumptions by eschewing the use of classical thermodynamics, never mentioning entropy, and using purely mechanical arguments. In this way we can identify c as the sound speed component of the characteristic speed and proceed to cover our whole subject. Although we will continue to speak of the entropy because it is conceptually useful, but it is perhaps well to remember that it is in no way necessary to the solution of our equations. It appears that this point is sometimes missed, with entropy introduced as a necessary fundamental quantity at the outset. See for example Truesdell[3].

Returning to our analysis, the next step is to eliminate ρ from (1.1a) in favor of p via (6.2). When this is done our system becomes

$$\dot{p} + \rho c^2 u_x = 0 \tag{6.4a}$$
$$\dot{u} + v p_x = 0 \tag{6.4b}$$
$$\dot{p} = c^2 \dot{\rho} \tag{6.4c}$$
$$c = c(\rho, s) \tag{6.4d}$$

Note that for each particle, c is some function of ρ with s as a parameter. We must of course know the value of s for each particle, and keep track of how it changes when the particle is shocked. Thus to actually compute with these equations as they stand is inconvenient, because we would have to track each particle by integrating

$$dx_p/dt = u \tag{6.5}$$

to find the particle paths $x_p(t)$ and thus determine $s(x,t)$. This strongly suggests use of a particle label instead of x as one independent variable. We shall not pursue this here, but instead continue with the characteristic analysis of the set (6.4).

Notice first that (6.4c) is already in characteristic form, since both derivatives are in the same direction, that of a particle path. We have then only to deal with (6.4a) and (6.4b). The analysis of these is the same as that of Chapter 5. The complete set of equations in characteristic form is

$$(dp/dt)_+ + z(du/dt)_+ = 0 \quad \text{on} \quad (dx/dt)_+ = u + c \tag{6.6a}$$
$$(dp/dt)_- - z(du/dt)_- = 0 \quad \text{on} \quad (dx/dt)_- = u - c \tag{6.6b}$$
$$(dp/d\rho)_0 = c \quad \text{on} \quad (dx/dt)_0 = u \quad . \tag{6.6c}$$

There are three families of characteristics: right and left running acoustic signals with velocity $u \pm c$, and particle paths with velocity u. As we saw earlier, entropy is constant on each particle path, by (6.2), so that on each particle path (away from shocks) c is a function of p or ρ alone, although it will in general be a different function of ρ on different particle paths. Because of this property, we were able to multiply (6.6c) by dt. We could not do this in (6.6a) and (6.6b), for these characteristics cross particle paths and thus z is in general not a function of p alone, but is instead a function of both p and s, with s in general a function of x and t. In the important special case of a constant-entropy flow, the entire set reduces to that of the Chapter 5, with z a function of p alone as before through the EOS $p = p(v;s)$, $s = \text{constant}$.

B. Shock Relations

We now turn to the jump conditions for the shock. We state without proof that these are the same as the equations relating any two points in a steady flow, and arrive at them by this route.

The first step is to write the equations in conservation form, in which each equation has the form

$$f_t + g_x = 0 \quad , \tag{6.7}$$

like (1.3) or (1.6a), and thus represents an integral conservation law like (1.2). In this form each equation has the form (6.7) with

(6.8)

	f	g
mass	ρ	ρu
momentum	u	$(\rho u)u + p$
energy	$\rho(e + 1/2\ u^2)$	$\rho(e + 1/2\ u^2)u + pu$

We obtain the equations for steady flow by simply setting the time derivatives to zero to obtain

$$\begin{aligned} (\rho u)_x &= 0 \\ ((\rho u)u + p)_x &= 0 \\ (\rho(e + 1/2\ u^2)u + pu)_x &= 0 \end{aligned} \tag{6.9}$$

These can of course be integrated immediately to give

$$(\rho u)_1 = (\rho u)_2 \tag{6.10}$$

etc.

where 1 and 2 are any two stations in the steady flow, or, as stated above, the states before and behind a shock (here u must be interpreted as the particle velocity in a frame in which the shock is at rest). These can be transformed into (exercise for the student) the usual forms (in the frame in which the material ahead of the shock is at rest)

$$\begin{aligned} u^2 &= (p - p_0)(v_0 - v) \\ \rho_0^2 D^2 &= (p - p_0)/(v_0 - v) \\ e - e_0 &= (p + p_0)(v_0 - v) \quad . \end{aligned} \tag{6.11}$$

We have already met the first two in Chapter 5. With use of the EOS p(v,e) to eliminate e, the third yields the Hugoniot curve in p-v, the locus of all possible shock states from a given initial state. The topology of the p-v plane for shocks is similar to that presented in Chapter 5 except that this Hugoniot curve replaces the EOS used there as the locus of shocked states. The Rayleigh line and constant-u curves are unchanged.

C. Rarefactions and Simple Waves

We still have simple waves, defined exactly as before in Chapter 5, and just as simple as before. The constant state adjacent to the simple wave must now of course have constant entropy as well as constant p and u, so that the entropy is constant throughout the simple wave. Consequently the equations of Chapter 5 apply with z(p) and c(p) replaced by z(p; s) and c(p; s), with s the given constant entropy. for example, the γ-law gas isentrope reads, as in (5.3)

$$p = k(s)\rho^\gamma \quad ,$$

the value of s affecting only the constant multiplier.

Thus, simple-wave rarefactions, including the centered rarefaction, are just like those of Chapter 5. But the moment a shock enters the picture, as when the rarefaction wave overtakes a shock, all is lost, for the flow then becomes one of varying entropy.

D. . . .Compressions

We can also have simple compression waves as before, up to the time a shock forms. Up to this time the description is again the same as that of Chapter 5. After shock formation, we have the same variable-entropy complication.

E. . . .Shocks

As stated earlier, the Hugoniot relations governing shocks are a new thermodynamic function, distinct from isentropes. They also have a different type of parameter: the initial state (that of the particle just before it enters the shock). The topology of the p-v plane with its Rayleigh line and constant-u loci is the same as in Chapter 5, except that the EOS curve is replaced by the Hugoniot curve through the initial state.

We shall compare Hugoniots and isentropies briefly, mostly by stating some properties. As suggested by Fig. 6.1, entropy increases monotonely with p or ρ along a Hugoniot. The isentrope through the initial state of a Hugoniot is second-order tangent to the Hugoniot, so that the isentrope is a reasonable approximation to the Hugoniot up to a reasonable shock strength. The "weak-shock approximation" based on this property is often very useful.

We may also compare the energy differential for an isentrope

$$de = -p\ dv \quad ,$$

with the energy difference for a Hugoniot.

$$\Delta e = -\bar{p}\ \Delta v \quad ,$$

with p the mean of the initial and final pressures. The ordinary differential equations for the two curves are also similar. For an isentrope in p-v we have

$$(dp/dv)_S = (p + e_v)/e_p \quad ,$$

and for the Hugoniot (sub H) (exercise for the student)

$$(dp/dv)_H = (\bar{p} + e_v)/(e_p - \tfrac{1}{2}\Delta v) \quad .$$

The Hugoniot lies everywhere above the isentrope through its initial state in p-v, and everywhere below it in p-u. An example for a γ-law gas is shown in Fig. 6.2.

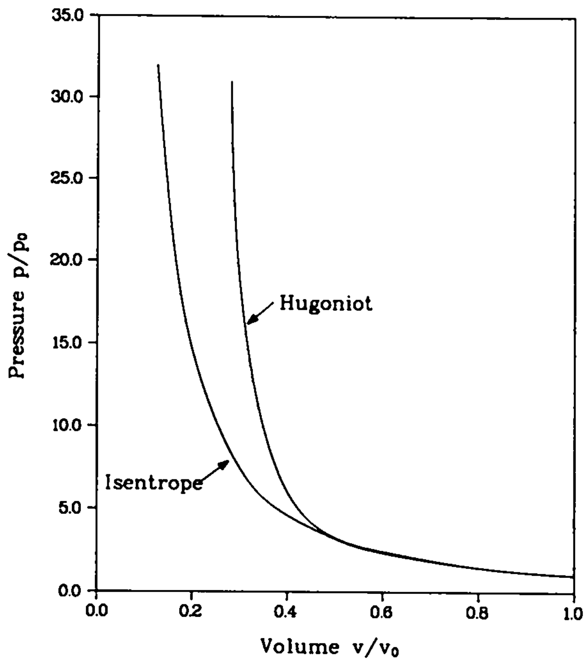


Fig. 6.2. Hugoniots and isentropes in p–v and p–u for a γ–law gas.

<u>1. Problem 6.1. Shock Reflection.</u> For the γ-law equation of state

$$e = pv/(\gamma - 1) \quad ,$$

derive the expression for the shock Hugoniot in p/p_0 vs. v/v_0, and p/p_0 vs. u^2/p_0v_0. A convenient abbreviation is $k = (\gamma - 1)/(\gamma + 1)$. Also derive expressions for p(u) and v/v_0 as a function of p in the strong shock limit $p/p_0 \rightarrow \infty$.

Calculate the ratio of reflected-shock to incident-shock pressure for shock reflection at a rigid wall for $\gamma = 1.4$ and $\gamma = 1.3$ for two incident-shock strengths: $p/p_0 = 2$ and $p/p_0 = \infty$.

<u>2. Problem 6.2. Murnaghan (Modified Tait) EOS.</u> Derive the energy expression e(p,v) which will give the Tait isentrope of problem 5.7. Choose the constant of integration so that e = 0 at $p_0 = 0$, $v = v_0$. Derive the expression for the Hugoniot curve in p - v through p = 0, $v = v_0$.

<u>3. Problem 6.3. Free-Surface Velocity.</u> Let a flat-topped shock run into a free surface. Using the Murnaghan equation of state with a = 1, $\gamma = 3$,

$v_0 = 1$, calculate the ratio of free-surface velocity u_{fs} to shock particle velocity u_p for shocks with $v/v_0 = 0.7$ and 0.6.

VII. NOMENCLATURE

A. Symbols

e - specific internal energy

D - shock velocity

p - pressure (or mass flux ρu for kinematic waves)

R - Riemann invariant

ρ - density

s - specific entropy

t - time

u - particle velocity

x - position

B. Terms

EOS - equation of state

piston - the (rear) boundary condition

C. Subscripts

o - initial state

+, -, 0 - forward and backward acoustic- and particle-path characteristics

overbar - unperturbed state

D. Differentiation

The usual subscript notation is used. For f(x,y)

$$f_x = \partial f/\partial x \quad , \quad f_y = \partial f/\partial y \quad .$$

A prime denotes ordinary differentiation. For f(x)

$$f' = df/dx \quad .$$

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